The discrete Painlevé II equations: Miura and auto-Bäcklund transformations

2003 J. Phys. A: Math. Gen. 36 8419

(http://iopscience.iop.org/0305-4470/36/31/305)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 8419–8431 PII: S0305-4470(03)63986-7

The discrete Painlevé II equations: Miura and auto-Bäcklund transformations

A S Carstea[1,3], **A Ramani**[1], **R Willox**[2,4] **and B Grammaticos**[2]

[1] CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France
[2] GMPIB, Université Paris VII, Tour 24-14, 5[e] étage, case 7021, 75251 Paris, France

Received 27 May 2003
Published 23 July 2003
Online at stacks.iop.org/JPhysA/36/8419

Abstract
We present Miura transformations for all discrete Painlevé II equations known to date. We then use these Miuras to derive special solutions in terms of discrete Airy functions and to construct auto-Bäcklund transformations for the discrete Painlevé equations. These transformations are then used to generate rational solutions. Some new forms of d-P_{II} and d-P_{34} are obtained as well.

PACS numbers: 02.30.Hq, 02.30.Ik

1. Introduction

The Painlevé equations are reputed for the richness of their interrelations. This is all the more true if we consider not just the six equations related to Painlevé transcendents but all the equations of the Painlevé/Gambier classification [1, 2]. A most interesting feature is the fact that some of these interrelations are just reductions of relations that hold between some well-known integrable evolution equations. The most famous among these is the transformation relating the KdV to the mKdV equation, also known as the Miura transformation [3] from the name of its inventor. Starting from

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

we introduce the transformation

$$u = -v^2 - v_x$$

for which we obtain

$$u_t + 6uu_x + u_{xxx} = 0.$$

[3] Permanent address: Institute of Physics and Nuclear Engineering, Department of Theoretical Physics, Magurele, Bucharest, Romania.
[4] Permanent address: Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.

Next we consider the similarity reduction of those equations. We introduce the new variables $v(x,t) = (3t)^{-1/3}w(z)$, $u(x,t) = (3t)^{-2/3}y(z)$, $z = x/(3t)^{1/3}$ and obtain the Miura transformation (a prime denotes $\mathrm{d}/\mathrm{d}z$)

$$y = -w' - w^2$$

and the equations

$$w'' = 2w^3 + zw + \kappa \tag{1.1}$$

$$q'' = \frac{q'^2}{2q} - q^2 - qz - \frac{(2\kappa+1)^2}{2q} \tag{1.2}$$

where $q = 2y - z$. Equation (1.1) is just the Painlevé II equation, while equation (1.2) is what is known as P_{34} (being the 34th equation in the Painlevé/Gambier classification) [4].

At this point we cannot resist the temptation of a remark on nomenclature, related to the names of the Painlevé equations. From the analogy with the KdV–mKdV one could consider that equation P_{II} is just the modified version of P_{34}. Still P_{II} is the more fundamental object and thus the use of the qualifier 'modified' would have been rather awkward in this case.

From the above analysis one could think that such Miura transformations have an asymmetrical structure leading from equation A to equation B but not the reverse. This erroneous conclusion is due to the fact that in most cases only one half of the Miura transformation is presented. The correct approach consists in introducing a pair of Miuras which relate the two equations in a completely symmetrical way. This is best assessed within the Hamiltonian formalism of Okamoto [5] for the description of Painlevé equations ($\mathbb{P}$s). Okamoto introduced the Hamiltonian for a given $\mathbb{P}$ and writes a Miura as the Hamiltonian equations of motion,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial p} \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial x}$$

where t is a 'time' related in a simple way to the independent variable of the $\mathbb{P}$. Eliminating either x or p between the two equations then yields a Painlevé equation and its Miura transformed equation. In the case of the Painlevé II equation the two Miura transforms are the following:

$$q = -2(w' + w^2) - z \tag{1.3}$$

$$w = \frac{q' + 2\kappa + 1}{2q}. \tag{1.4}$$

One particularly useful application of the Miura transformation, apart from generating other equations, is the derivation of elementary solutions for the Painlevé equation [6]. Indeed by putting $\kappa = -1/2$ in the Painlevé II equation (1.1) P_{34} (1.2) has a special solution $q = 0$ which makes w in (1.4) become indeterminate. But in this case the indeterminacy can be lifted using the first Miura (1.3). Indeed the latter becomes

$$z/2 = -w' - w^2$$

which can be linearized using $w = A'/A$ to the Airy equation

$$A'' + (z/2)A = 0. \tag{1.5}$$

Another use of the Miura transformations is to generate auto-Bäcklund (or Schlesinger) transformations for the initial equation. This is done through the combination of the Miura and some invariance of the transformed equation. In the case at hand, we remark that P_{34} is invariant under the transformation $2\kappa + 1 \to -(2\kappa + 1)$. Moreover, P_{II} is invariant with respect to a simultaneous change in sign of the dependent variable and the parameter. Thus we start from $w(\kappa)$, use (1.3) to obtain q and using the Miura (1.4) we construct

$$u = \frac{q' - (2\kappa + 1)}{2q} \tag{1.6}$$

which would be a solution of P_{II} with parameter $-(1+\kappa)$. Introducing $v = -u$ we obtain a solution of P_{II} with parameter $\kappa + 1$. Using (1.3), (1.4) and (1.6) and eliminating q we can derive the auto-Bäcklund transformation (which is indeed a Schlesinger),

$$v = -w - \frac{\kappa + 1/2}{w' + w^2 + z/2} \tag{1.7}$$

where $w \equiv w(\kappa)$ and $v \equiv w(\kappa + 1)$. One particular use we have for the auto-Bäcklund/Schlesinger transformations is to generate special solutions starting from an elementary, 'seed', one. This is best illustrated in the case of the rational solutions of P_{II}. Remarking that when $\kappa = 0$, P_{II} has a solution $w = 0$, we can use (1.7) to construct 'higher' rational solutions. We thus find for $\kappa = 1$ the solution $w = -1/z$; for $\kappa = 2$, $w = (4 - 2z^3)/(4z + z^4)$ and we can iteratively construct solutions for any integer κ.

In this paper, we shall transpose the remarks above to the case of discrete Painlevé equations. This work has been motivated by recent results of ours [7] where we have investigated the possible discrete forms of the Painlevé II equations and obtained eight different such equations [8]. In what follows, we shall present their Miura transformations and identify the equations that the d-$\mathbb{P}$s are transformed into. Using the Miuras we shall also derive the elementary solutions of all these d-P_{II}s in terms of discrete Airy functions. We shall also construct the auto-Bäcklund transformations for these d-P_{II}s and present their first few rational solutions. Some new forms of d-P_{II} will also be derived.

2. Constructing Miura and auto-Bäcklund transformations

The first question that springs to mind when one considers the problem of finding Miura transformations for d-$\mathbb{P}$s is how one can construct them in a systematic way. In previous publications, we have shown how the use of the bilinear formalism can be of considerable help [9]. However, this intermediate step based on τ functions is not mandatory. One can proceed to a direct construction of a discrete Miura transformation starting from the simple observation that in the continuous case the Miuras are ratios of Riccati-like quantities $\alpha u' + \beta u^2 + \gamma u + \delta$. Thus in the discrete case we introduce the following general ansatz of the Miura,

$$x_n = \frac{\alpha y_n y_{n+1} + \beta y_{n+1} + \gamma y_n + \delta}{\epsilon y_n y_{n+1} + \zeta y_{n+1} + \eta y_n + \theta} \tag{2.1a}$$

$$y_n = \frac{a x_n x_{n-1} + b x_{n-1} + c x_n + d}{e x_n x_{n-1} + f x_{n-1} + g x_n + h} \tag{2.1b}$$

where $\alpha, \beta, \ldots, a, b, \ldots, h$ are functions of the independent variable. Thus the problem of the construction of the Miura given a discrete $\mathbb{P}$,

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_4(x_n) - x_{n-1} f_3(x_n)} \tag{2.2}$$

(where f_i are in general quartic polynomials in x_n) consists in determining the functions $\alpha, \ldots, h$ and deriving the equations for y. As a matter of fact, the general form of the auto-Bäcklund transformation is also the same as (2.1). An equation like (2.1b) would, for example, represent an auto-Bäcklund transformation where x and y are solutions of the same d-$\mathbb{P}$ associated with different sets of parameters.

We can illustrate this in the case of the standard d-P_{II}. We start from the mapping

$$x_{n+1} + x_{n-1} = \frac{x_n(z_n + z_{n-1}) + \mu + \alpha}{x_n^2 - 1} \tag{2.3}$$

where $z_n = \alpha n + \beta$. The Miura obtained in [10] is

$$y_n = (x_n - 1)(x_{n+1} + 1) - z_n \tag{2.4a}$$

$$x_n = \frac{y_{n-1} - y_n + \mu}{y_n + y_{n-1}}. \tag{2.4b}$$

Eliminating x_n leads to the discrete form of P_{34},

$$(y_{n+1} + y_n)(y_{n-1} + y_n) = \frac{-4y_n^2 + \mu^2}{y_n + z_n}. \tag{2.5}$$

Obtaining the Airy solutions for (2.3) is straightforward. Putting $y_n = 0$ and $\mu = 0$ we find from (2.4a)

$$x_{n+1} = \frac{z_n + 1 - x_n}{x_n - 1} \tag{2.6}$$

which is a discrete Riccati that linearizes through a Cole–Hopf transformation $x_n = 1 + R_n/R_{n-1}$ and we thus find that (2.6) linearizes to a discrete analogue of the Airy equation:

$$R_{n+1} - 2R_n + z_n R_{n-1} = 0. \tag{2.7}$$

The auto-Bäcklund transformation for (2.3) has the form

$$\chi_n = -x_n - \frac{\mu(x_n - 1)}{(x_{n+1} + 1)(x_n - 1) - z_n - \mu/2} \tag{2.8}$$

where $\chi_n = x_n(\mu - 2\alpha)$ and $x_n = x_n(\mu)$ correspond to parameters $\mu - 2\alpha$ and μ respectively. The auto-Bäcklund transformation (2.8) can be used to generate solutions starting from a given 'seed'. In the case of (2.3) we have a rational solution $x_n = 0$ when $\mu = -\alpha$. Using (2.8) we find that $\chi_n = \alpha/(1 + z_n - \alpha/2)$ is another rational solution which exists for $\mu = -3\alpha$.

A second discrete Painlevé equation is the one we call the alternate d-P_{II}

$$\frac{z_n}{x_{n+1}x_n + 1} + \frac{z_{n-1}}{x_{n-1}x_n + 1} = -x_n + \frac{1}{x_n} + z_n + k. \tag{2.9}$$

This equation was extensively studied in [11] and its Miura (obtained in [12]) was investigated in considerable detail in [13]. We shall not go into these details here lest we overburden this paper with results which have already been known for quite a few years.

3. The Miura and auto-Bäcklund transformations of the various discrete Painlevé II equations

Six new d-P_{II} equations were analysed in [7] from the point of view of the bilinear formalism. Their full (parameter) freedom was also investigated there, which led us to conclude that most of them (with just one exception) are actually reduced forms of systems that are far richer than a mere d-P_{II}. Still the interesting fact remains that these 'reduced' d-P_{II} forms possess all the nice properties one would expect from any discrete analogue of the Painlevé II equation. In what follows, we shall examine these equations as d-P_{II}s, i.e., we shall produce their Miura transformations, derive the associated d-P_{34}, obtain elementary Airy-type solutions under the appropriate constraints, derive their auto-Bäcklund transformations and use these to obtain rational solutions for the d-$\mathrm{P_{II}}$s starting from (trivial) 'seed' solutions.

(i) The first mapping we shall examine is a q-$\mathrm{P_{II}}$ equation of the form

$$x_{n+1}x_{n-1} = \frac{a_n(x_n - b_n)}{x_n(x_n - 1)}. \tag{3.1}$$

The full freedom of a_n and b_n, that is compatible with integrability, as presented in [14] is given by $\log a_n = \alpha n + \beta + \gamma(-1)^n$ and $\log b_n = \alpha n + \delta - \gamma(-1)^n$ but in what follows we shall work with $\gamma = 0$ and introduce $b_n = q_n = \mathrm{e}^{n\alpha} \equiv q_0\lambda^n$, $a_n = aq_n$. The Miura transformation for (3.1) is

$$y_n = \frac{c}{aq_n}x_{n-1}(x_n - 1) \tag{3.2a}$$

$$x_n = cq_n\frac{y_{n+1}y_n - 1}{y_n - c} \tag{3.2b}$$

where $c^2 = -a\lambda$ is a constant. From (3.2) we obtain the q-$\mathrm{P_{34}}$ equation

$$(y_ny_{n+1} - 1)(y_ny_{n-1} - 1) = \frac{1}{q_n}(y_n - c)(y_n - 1/c). \tag{3.3}$$

In order to find the Airy-type solutions the indeterminacy of x_n in rhs of (3.3b) appears when $y_n = c$ and $c^2 = 1$. In this case, i.e. $a\lambda = -1$ we have a solution of (3.1) given by

$$x_{n-1}(x_n - 1) = -q_{n-1}. \tag{3.4}$$

Putting $x_n = R_{n+1}/R_n$ we can linearize (3.4) to

$$R_{n+1} - R_n + q_{n-1}R_{n-1} = 0$$

which is a q-discrete form of the Airy equation. This can be easily checked through the continuous limit of (3.4). Using the ansatz introduced in [8] $x_n = (1 + \epsilon w)/2, \lambda = 1 + \epsilon^3/2, q_0 = 1/4$, we find that (3.4) reduces to the Ricatti $w' = -w^2 - z/2$, while the condition $a\lambda = -1$ goes over to $\kappa = -1/2$. Similarly, the ansatz $y_n = 1 - \epsilon^2 u$ leads to $\mathrm{P_{34}}$ for u at the continuous limit.

In order to construct the auto-Bäcklund transformations we shall use the invariance of q-$\mathrm{P_{34}}$ under the transformation $c \to 1/c$. Thus one starts from (3.2a) where $c \to 1/c$ (but with the same y_n) and uses (3.2a) to eliminate y_n. In the resulting equation we use q-$\mathrm{P_{II}}$, to eliminate x_{n-1} and obtain finally

$$X_n = \frac{x_n(x_{n+1} - 1)(x_n - q_n) + q_nx_nx_{n+1}}{x_nx_{n+1} + a\lambda(x_n - q_n)} \tag{3.5}$$

which is a solution of q-$\mathrm{P_{II}}$ for parameter $1/a\lambda^2$. Now, using the invariance of (3.1) under $x_n \to q_n/x_n$, $a \to 1/a$ we construct $\chi_n = q_n/X_n$ which is indeed a Schlesinger transformation since χ_n is a solution corresponding to parameter $\tilde{a} = a\lambda^2$.

The rational solutions of q-$\mathrm{P_{II}}$ can be constructed starting from the seed solution $x_n = \sqrt{q_n}$ which exists provided $a = -1$. Using the auto-Bäcklund (3.5) we find that for $a = -1/\lambda^2$ we have the next rational solution

$$x_n = c\sqrt{q_n}\frac{c - (c+1)\sqrt{q_n}}{1 - (c+1)\sqrt{q_n}} \tag{3.6}$$

where $c = 1/\sqrt{\lambda}$.

(ii) The second mapping is again a q-$\mathrm{P_{II}}$

$$x_{n+1}x_{n-1} = \frac{a_n(x_n - b_n)}{x_n - 1} \tag{3.7}$$

where the full freedom of a_n and b_n is given by $\log a_n = \alpha n + \beta + (-1)^n\gamma + j^n\delta + j^{2n}\zeta$, $\log b_n = 2\alpha n + \theta - (-1)^n\gamma - j^n\delta - j^{2n}\zeta$ where $j^3 = 1$. In what follows, we shall restrict ourselves to $\gamma = \delta = \zeta = 0$ and choose the origin of n such that $a_n = aq_n$, $b_n = q_n^2$ with $q_n = \lambda^n$.

The Miura transformation of (3.7) is

$$x_n = \frac{a_{n+1}(1 - cy_n)}{c(y_{n-1} - c)} \tag{3.8a}$$

$$y_n = \frac{c\left(x_n x_{n+1} - x_n + q_n^2\right)}{a_{n+1}x_{n+1}} \tag{3.8b}$$

where $c = a\mu^3$ with $\lambda = \mu^2$. Eliminating x_n we find the q-P_{34}

$$(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = \left(1 - \frac{y_n}{a\mu}\right)(y_n - c)\left(y_n - \frac{1}{c}\right). \tag{3.9}$$

The Airy-type solution for (3.7) is obtained from (3.8b) with $y_n = c$ provided $c^2 = 1$, i.e. $a^2\lambda^3 = 1$. This leads to the discrete Riccati

$$x_{n+1}(x_n - a_{n+1}) = x_n - b_n \tag{3.10}$$

which can be linearized through $x_n = 1 + a_n R_n/R_{n+1}$ to

$$(a_{n+1} - 1)R_{n+1} - (a_n - 1)R_n + a_{n-1}R_{n-1} = 0$$

which is not a mere q-Airy, but in fact a q-confluent hypergeometric equation. However, it can be seen that the continuous limit of (3.10) is $w' = -w^2 - z/2$, as expected, provided that $\lambda = 1 + 2\epsilon^3/3$, $x_n = 1/3 + 2\epsilon w/3$.

For the construction of the auto-Bäcklund we remark that (3.9) is invariant under $c \to 1/c$ while (3.7) is invariant under $x_n \to q_n^2\big/x_n$, $a \to 1/a$. We find finally that

$$X_n = \frac{q_n^2(a_{n+1}a_{n+2} - a_{n+1}x_{n+1} - c^2 x_n + x_n x_{n+1})}{x_n\left(q_n^2 - a_{n+1}x_{n+1} + x_n x_{n+1} - x_n\right)} \tag{3.11}$$

is a Schlesinger transformation, i.e., X_n is a solution of q-$\mathrm{P_{II}}$ with parameter $\tilde{a} = a\lambda^3$.

Rational solutions of q-$\mathrm{P_{II}}$ (3.7) can be constructed starting from the seed solution $x_n = \mp q_n$ which exists for $a = \pm 1$. From this solution we can construct, using (3.11), the higher ones. We have for instance

$$x_n = \mp q_{n+1}\frac{q_n(\lambda^2 + \lambda + 1) \pm \lambda^2}{q_n(\lambda^2 + \lambda + 1) \pm 1}$$

which satisfies q-$\mathrm{P_{II}}$ for the parameter $\pm\lambda^3$.

(iii) The third mapping is

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = a_n x_n/(x_n - b_n) \tag{3.12}$$

which is a genuine q-$\mathrm{P_{II}}$ equation with the parameters $a_n = aq_n^2$, $b_n = q_n$ with $q_n = \lambda^n$. The Miura transformations are given by

$$y_n = \frac{\rho_n x_n}{x_n x_{n+1} - 1} \tag{3.13a}$$

$$x_n = \frac{\rho_{n-1}y_n - q_n}{y_n y_{n-1} - 1} \tag{3.13b}$$

where $a_n = \rho_n\rho_{n-1}$. Eliminating x_n and introducing the constant $c^2 = \lambda/a$ we obtain the corresponding q-P_{34}

$$(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = -\frac{q_n\rho_n}{y_n}(y_n - c)(y_n - 1/c). \tag{3.14}$$

An Airy-type solution can be found in the case $a = \lambda$ and $y_n = 1$. Using the equation (3.13a) we get the discrete Riccati equation

$$x_n x_{n+1} - 1 - q_{n+1}x_n = 0$$

which can be linearized by means of $x_n = R_{n+1}/R_n$ to the q-Airy equation

$$R_{n+1} - q_n R_n - R_{n-1} = 0.$$

In order to construct the Schlesinger transformation we use the symmetry of q-P_{34}, $c \to 1/c$, which leads to the following auto-Bäcklund:

$$X_n = \frac{c(q_n - x_n)(x_n x_{n-1} - 1) + q_n/c}{q_n x_{n-1} - x_n x_{n-1} + 1}. \tag{3.15}$$

The variable X_n satisfies an equation of the form (3.12) where both a_n and b_n are modified. We find indeed that the new parameter is $A = \lambda^2/a$ provided we choose as independent variable $Q_n = \rho_{n-1}$ (which means that we must have $B_n = \rho_{n-1}$).

In order to obtain the Schlesinger transformation we use the self-Miura transformation given by

$$\chi_n = -\frac{\rho_{n-1}}{x_n x_{n-1} - 1} \tag{3.16a}$$

$$x_n = -\frac{q_n}{\chi_n \chi_{n+1} - 1} \tag{3.16b}$$

where χ_n verifies the same mapping

$$(\chi_n\chi_{n+1} - 1)(\chi_n\chi_{n-1} - 1) = \tilde{a}_n\chi_n/(\chi_n - \tilde{b}_n)$$

with $\tilde{a}_n = q_n q_{n-1}$ and $\tilde{b}_n = \rho_{n-1}$, i.e. $\tilde{a} = 1/a$. Combining (3.15) and (3.16) one obtains the true Schlesinger transformation corresponding to a parameter a/λ^2 but the explicit expression is too lengthy to be given here.

(iv) The fourth mapping is

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = a_n(x_n - b_n). \tag{3.17}$$

The full freedom of parameters is given by $\log a_n = 3\alpha n + \beta$, $\log b_n = -\alpha n + \gamma + j^n\delta + j^{2n}\zeta$. We shall restrict ourselves to the symmetric case $\delta = \gamma = 0$ so that we have a q-P_{II} equation. We rewrite the mapping in a more convenient way,

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = a q_n^3 x_n + q_n^2 \tag{3.18}$$

where $q_n = \lambda^n$. The self-Miura transformation is given by

$$\chi_n = \frac{\mu q_n(a q_n x_n + 1)}{(x_n x_{n+1} - 1)} \tag{3.19a}$$

$$x_n = \frac{\rho_n(1 + \rho_n\chi_n/a)}{\mu(\chi_n\chi_{n-1} - 1)} \tag{3.19b}$$

where $\lambda = \mu^2$ and $\rho_n = a q_n/\mu$. The equation for χ_n is

$$(\chi_n\chi_{n+1} - 1)(\chi_n\chi_{n-1} - 1) = \frac{\rho_n^3}{a}\chi_n + \rho_n^2 \tag{3.20}$$

so $q_n \to \rho_n$ and $a \to 1/a$. This is not a Schlesinger transformation. Its effect is to invert the parameter a and its square is just a downshift, i.e. $x_n \to \chi_n \to x_{n-1}$.

The actual Miura transformation is given by

$$x_n = -\frac{q_n(a^2 y_n - s^3)}{as(y_n y_{n-1} - 1)} \tag{3.21a}$$

$$y_n = -\frac{\lambda q_n(a x_n + q_n s^2)}{s(x_n x_{n+1} - 1)} \tag{3.21b}$$

where $s = a\mu$. Eliminating x_n we find the q-P$_{34}$ equation

$$(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = \frac{(s^3 y_n - a^2)(a^2 y_n - s^3)}{a^2 s^3\big(1 - y_n/\big(\lambda q_n^2 s\big)\big)}. \tag{3.22}$$

The Airy-type solutions are given by linearization of the discrete Riccati equation

$$a(x_n x_{n+1} - 1) + \mu q_n(a x_n + q_n s^2) = 0 \tag{3.23}$$

which is obtained from (3.21b) provided that $s^3 = a^2$, i.e. $a = 1/\mu^3$ and $y = 1$. The q-Airy equation is (putting $x_n = R_n/R_{n-1}$) given by

$$R_{n+1} + \mu q_n R_n + (q_n q_{n-1} - 1)R_{n-1} = 0.$$

In order to construct the auto-Bäcklund transformation we write the q-P$_{34}$ equation in a simpler way,

$$(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = \frac{(y_n - c)(y_n - 1/c)}{\big(1 - y_n/\big(q_n^2 c\big)\big)} \tag{3.24}$$

where $c = s^3/a^2 = a\mu^3$. Now we are using the symmetry $c \to 1/c$ which is valid provided that q_n also is changing so that $q_n^2 c \to q_n^2\big/c$. We introduce these new values in the Miura transformation (3.21a) and obtain

$$\chi_n = \frac{q_n}{\mu}\frac{1 - y_n c}{y_n y_{n-1} - 1}$$

which is a solution of

$$(\chi_n \chi_{n-1} - 1)(\chi_n \chi_{n+1} - 1) = A Q_n^3 \chi_n + Q_n^2 \tag{3.25}$$

where $A = 1/a\lambda^3$, $Q_n = \rho_{n+2}$. It suffices now to apply the self-Miura (3.19b) to obtain

$$\xi_n = \mu Q_n \frac{1 + Q_{n+1}\chi_{n+1}/A}{\chi_n \chi_{n+1} - 1}$$

which satisfies (3.18) with parameter $\tilde{a} = a\lambda^3$. Thus, ξ_n indeed introduces a Schlesinger transformation (the explicit form being too lengthy to be written here).

(v) The fifth mapping is

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = (1 - a_n x_n)(1 - b_n x_n). \tag{3.26}$$

The full freedom of parameters is $\log a_n = \alpha n + \beta + (-1)^n(n\gamma + \delta) + \mathrm{i}^n\zeta + (-\mathrm{i})^n\eta$, $\log b_n = \alpha n + \theta + (-1)^n(n\gamma - \delta) - \mathrm{i}^n\zeta - (-\mathrm{i})^n\eta$. Here we shall restrict ourselves to $a_n = a q_n$, $b_n = q_n/a$ where $q_n = q_0\lambda^n$. The continuous limit of the equation is $w'' = 2w^3 + wz + \kappa$ provided that $x_n = \epsilon\sqrt{2}w$, $\lambda = 1 + \epsilon^3/4$, $q_0 = \mathrm{i}\sqrt{2}$, and $a = \mathrm{i}(1 - \epsilon^3\kappa/2)$.

The self-Miura transformation in this case is very simple (being in fact a downshift of x_n),

$$\chi_n = \frac{q_n^2 x_n - (a + 1/a)q_n + x_{n+1}}{x_n x_{n+1} - 1} \equiv x_{n-1}.$$

The Miura transformations are given by

$$x_n = \frac{q_n(c(y_{n-1}-1)+y_n-1)}{a\lambda(y_n y_{n-1}-1)} \tag{3.27a}$$

$$y_n = \frac{-q_n q_{n+1} + a q_{n+1} x_{n+1} + q_n x_n/a - 1}{x_n x_{n+1} - 1} \tag{3.27b}$$

with $c = a^2\lambda^2$. Eliminating x_n we find the q-P_{34}

$$(y_n y_{n+1}-1)(y_n y_{n-1}-1) = \frac{q_n q_{n+1}(y_n-1)(y_n-c)(y_n-1/c)}{(y_n - q_n q_{n+1})}. \tag{3.28}$$

One can see that $y_n = 1$ is always a solution of the q-P_{34} equation and in this case x_n becomes indeterminate in the Miura (3.27a). From the second Miura we get that $x_n x_{n+1} - 1 = -q_n q_{n+1} + a q_{n+1} x_{n+1} + q_n x_n/a - 1$ which factorizes to

$$(x_n - a q_{n+1})(x_{n+1} - q_n/a) = 0 \tag{3.29}$$

which leads to rational solutions. This solution is compatible with (3.26) only if $a^2\lambda^2 = 1$. However, this constraint cannot be satisfied in the sector of the continuous limit we described above. The present rational case is interesting nonetheless as it allows us to construct a linearizable solution to (3.26). Starting from $a = \epsilon/\lambda$ (where $\epsilon^2 = 1$) we find that equation (3.29) becomes $(x_n - \epsilon q_n)(x_{n+1} - \epsilon q_{n+1}) = 0$. Since this relation is a product for two consecutive n one could imagine that the solution $x_n = \epsilon q_n$ is valid only for even or for odd n. If we consider (3.26) for, say, $n = 2m+1$ it is straightforward to show that $x_{2m} = \epsilon q_{2m}$ leads to an identity and that $x_{2m+1} = \epsilon q_{2m+1}$ leads to an equation for x_{2m+2} that is homographic in terms of x_{2m} : $x_{2m+2}x_{2m} - \epsilon(x_{2m+2}+x_{2m})/q_{2m+1} - q_{2m+1}^2 + \lambda + 1/\lambda = 0$. This equation can be linearized to a second-order mapping with a Cole–Hopf transformation but in this case the situation is even better since $x_{2m} = \epsilon q_{2m}$ is a solution. Using this solution it is straightforward to reduce this equation to a linear *first-order* mapping. Putting $x_{2m} = \epsilon q_{2m} + 1/R_{2m}$ we find for R the (inhomogeneous) linear equation:

$$(q_{2m+2}q_{2m+1}-1)R_{2m+2} + (q_{2m+1}q_{2m}-1)R_{2m} + \epsilon q_{2m+1} = 0.$$

On the other hand, if $c = a^2, \lambda^2 = -1$ then $y_n = -1$ is also a solution. This solution gives a different linearizable case which, as a matter of fact, is the only one found in the sector of the continuous limit. The Airy-type solutions arise from the Riccati-type equation $x_n x_{n+1} - q_n q_{n+1} + a q_{n+1} x_{n+1} + q_n x_n/a - 2 = 0$ which can be linearized through $x_n = -a q_{n+1} + R_{n+1}/R_n$ to

$$R_{n+1} + 2a q_{n+1} R_n - 2(q_n q_{n-1}+1)R_{n-1} = 0.$$

In order to construct the auto-Bäcklund/Schlesinger transformation we use the symmetry that consists in interchanging c and $1/c$ in the q-P_{34} equation. Finally, we obtain

$$X_n = \frac{x_n^2(a q_n - c x_{n-1}) + x_n(ac q_{n+1} x_{n-1} + c - 1 - q_n q_{n-1}) + q_{n-1}(1-c^2)/a}{x_n(ac q_n - x_{n-1}) + 1 + a q_{n+1} x_{n-1} - c(1 + q_n q_{n-1})} \tag{3.30}$$

which satisfies (3.26) with parameter $a\lambda^2$.

The simplest rational solution, $x_n = 0$, is obtained for $a = \pm\mathrm{i}$. The next rational solution is

$$x_n = \pm\mathrm{i}\frac{(\lambda^4-1)q_{n-1}}{1+\lambda^2(1+q_{n-1}q_n)}$$

which exists when $a = \pm\mathrm{i}/\lambda^2$.

(vi) The next mapping we shall treat is a difference equation,

$$(x_n + x_{n+1})(x_n + x_{n-1}) = \frac{(x_n + z_n + k_n)(x^2 - b^2)}{(x_n - 2z_n)}. \tag{3.31}$$

The full freedom of this equation is given by $z_n = \alpha n + \eta + j^n\gamma + j^{2n}\delta$, $k_n = \sigma + (-1)^n\theta - 3j^n\gamma - 3j^{2n}\delta$ and we are treating only the symmetric case $\gamma = \delta = \theta = 0$ which means that k_n is a constant. An interesting property of this mapping is that it possesses two different continuous limits. The first one is $x_n = 3 + 4\epsilon w$, $b = 9 + 8\epsilon^3\kappa/3$, $k = -9/2$ and $z_n = 7/2 - 4\epsilon^2 t$ which goes to a Painlevé II equation $w'' = 2w^3 + wt + \kappa$ and the other one is $x_n = -8\epsilon^2 w$, $b = 4\epsilon^3\kappa/3$, $k = 9$ and $z_n = -1 - 4\epsilon^2 t$ which leads to P_{34} equation $w'' = w'^2/(2w) + 2w^2 - wt - g^2/(2w)$.

So equation (3.31) is a discrete P_{II} and a discrete P_{34} at the same time. It is therefore not astonishing that it possesses a genuine auto-Miura,

$$y_n = z_n - m - \frac{(x_n - b)(x_{n+1} - z_n - k)}{x_n + x_{n+1}} \tag{3.32a}$$

$$x_n = z_{n-1} + k - \frac{(y_n - \beta)(y_{n-1} - z_{n-1} + m)}{y_n + y_{n-1}} \tag{3.32b}$$

where $2m = b - \alpha$ and $2\beta = b - 2k + 3\alpha$. Using (3.32) we obtain

$$(y_n + y_{n+1})(y_n + y_{n-1}) = \frac{(y_n + \zeta_n + \kappa)(y^2 - \beta^2)}{(y_n - 2\zeta_n)} \tag{3.33}$$

with $2\zeta_n = 2z_n + \alpha + k + m$ and $2\kappa = -(k + 3m + 3\alpha)$. We remark that (3.33) is invariant under $\beta \to -\beta$. We can use this property to obtain an auto-Bäcklund transformation for (3.31). The standard procedure is to start from a given x_n, use (3.32*a*) and construct y_n, then change the sign of β in (3.32*b*) and, *with the same* y_n, compute the new x_n. The latter, however, would not be a Schlesinger transform of the original x_n but of the y_n that could have been obtained from (3.32*a*) for the opposite value of b. To get a Schlesinger transform one would therefore need a third Miura. Hence, in order to work with only two Miuras rather than three, we choose not to proceed in this way. Rather, we start from (3.32*a*) and obtain y_n which, given the fact that x_n and y_n satisfy the same equation (although with different parameters) can be taken as a new x, $y_n \equiv \chi_n$. Then (3.32*a*), with $\chi_n, \zeta_n, \kappa, \beta$ and $2\mu = \beta - \alpha$ can be used to compute a new y. Calling this new variable X_n we find that it satisfies (3.31) with parameters $Z_n = z_n + \alpha/2$, $K = k - 3\alpha/2$ and $B = b + 3\alpha$. This defines an auto-Bäcklund/Schlesinger transformation for (3.31). The explicit form of this auto-Bäcklund transformation will not be given here since it would cover several lines. Still, the procedure outlined just above allows for its algorithmic construction and one can use the transformation thus obtained, for instance, to derive explicit solutions of linearizable or rational type.

In order to obtain the Airy-type solutions, we consider the Miura with $y_n = \beta = 0$ which means that $k = b/2 + 3\alpha/2$. From the transformation (3.32*a*) we find

$$(x_n + x_{n+1})(z_n - m) - (x_n - b)(x_{n+1} - z_n - k) = 0$$

which can be linearized through the substitution $x_n = z_{n+1} + m + (m - z_{n-1})R_n/R_{n-1}$ to the following discrete linear equation:

$$(R_{n+1} + R_n)(2z_n - b - \alpha) - 2R_{n-1}(2z_{n+1} + b) = 0. \tag{3.34}$$

In order to construct rational solutions one can see that if $b^2 = 4k^2$ then $x_n = -2k/3$ is an elementary rational solution. The next rational solution computed with the help of the auto-Bäcklund transformation is $x_n = 3z_n + k/3$, obtained for $b^2 = 9\alpha^2/4$. We will give one more rational solution which is a Schlesinger transform of the elementary one, obtained for $b^2 = 4(k+3\alpha)^2$, namely $x_n = -(\alpha + 2k/3)(9z_n + 7k)/(9z_n + 7k + 12\alpha)$.

If $k = 3\alpha/4$ then the elementary solution $x_n = -2k/3$ is also in the linearizable sector because one can take $b = -2k = -3\alpha/2$ for which the linearizability condition $k = b/2 + 3\alpha/2$ holds. What is remarkable here is that the condition for the existence of the second rational solution is also satisfied. Indeed, if we look at the linear equation (3.34) for $b = -3\alpha/2$ it simplifies to $R_{n+1} + R_n - 2R_{n-1} = 0$ with solution $R_n = P + (-2)^n Q$ where P and Q are constants. For $Q = 0$ we find that R is a constant and so $x_n = -\alpha/2 = -2k/3$. For $P = 0$, $R_n/R_{n-1} = -2$ so $x_n = 3z_n + \alpha/4 = 3z_n + k/3$. In fact we have an exact expression of x_n in terms of elementary functions, for a full family of values P/Q. This is the only case we know where such a situation arises for discrete Painlevé equations.

Finally, while investigating the possible forms of discrete $\mathrm{P_{II}}$ we concluded that another form should exist, beyond those obtained in [7]. The main indication for this fact was that there existed one discrete $\mathrm{P_{34}}$ equation, identified in [8], for which no corresponding d-$\mathrm{P_{II}}$ was found. It turned out that another q-$\mathrm{P_{II}}$ indeed exists, one which assumes the form

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{(1 - x_n/q_n)(1 - x_n q_{n+1})(1 - x_n q_{n-1})}{1 - a x_n \big/ q_n^3} \tag{3.35}$$

with $q_n \equiv q_0 \lambda^n$. The full freedom of the coefficients is $\log q_n = \alpha n + \beta + \gamma k^n + \delta k^{2n} + \zeta k^{3n} + \eta k^{4n}$ and $\log a_n = \theta + (2 - k^2 - k^3)(\gamma k^n + \eta k^{4n}) + (2 - k - k^4)(\delta k^{2n} + \zeta k^{3n})$ where k is a fifth root of unity. The geometry of this equation is described by the Weyl group $E_6^{(1)}$.

Putting $x = \mathrm{i}(3 + 5\epsilon w)$, $\lambda = 1 - \epsilon^3/4$, $q = \mathrm{i}(-2 + \epsilon^2 t/2)$, $a = 1 + 5\kappa\epsilon^3/2$, we obtain at the continuous limit the continuous $\mathrm{P_{II}}$: $w'' = 2w^3 + tw + \kappa$. Equation (3.35) possesses an self-Miura transformation of the form

$$\chi_n = \frac{\rho_n(q_n x_{n+1} + q_{n+1} x_n - q_n q_{n+1} - 1)}{x_n x_{n+1} - 1} \tag{3.36a}$$

$$x_n = \frac{q_n(\rho_n \chi_{n-1} + \rho_{n-1}\chi_n - \rho_n \rho_{n-1} - 1)}{\chi_n \chi_{n-1} - 1} \tag{3.36b}$$

where $\rho_n = q_n\sqrt{\lambda/a}$. Eliminating x between the two equations we obtain

$$(\chi_n \chi_{n+1} - 1)(\chi_n \chi_{n-1} - 1) = \frac{(1 - \chi_n/\rho_n)(1 - \chi_n \rho_{n+1})(1 - \chi_n \rho_{n-1})}{1 - \alpha \chi_n \big/ \rho_n^3} \tag{3.37}$$

which is precisely (3.35) for the variables χ, ρ and parameter $\alpha \equiv 1/a$.

However, (3.35) does also possess a genuine Miura,

$$y_n = \frac{p_n\big(m^2\lambda^4 x_{n-1} + p_n^2 x_n - m\lambda^3 p_n - m\lambda p_n\big)}{m\lambda^3(x_n x_{n-1} - 1)} \tag{3.38a}$$

$$x_n = \frac{p_n\big(b y_{n+1} + p_n^2 y_n - 1 - b p_n^2\big)}{m(y_n y_{n+1} - 1)} \tag{3.38b}$$

where $b = m^2\lambda$ and $p_n = m\lambda^2 q_n$, which allows one to recover on the one hand equation (3.35) with $a = 1/(m^4\lambda^7)$ and on the other hand the (previously known) q-$\mathrm{P_{34}}$:

$$(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = \frac{\big(1 - p_n^2 y_n\big)(y_n - b)(y_n - 1/b)}{\big(1 - y_n/p_n^2\big)}. \tag{3.39}$$

This is equation (6.7) of [8] (and what was called *there* $a \equiv q$ is now p_n^2).

Airy-type solutions can be found in the case $b \equiv m^2\lambda = 1$ (i.e. $a = \lambda^{-5}$) and $y_n = 1$. Putting $x_n = mp_{n+1} + \left(p_n^2 - 1\right)R_n/R_{n+1}$ we obtain the linear q-Airy equation:

$$R_{n+1} + mp_n R_n - R_{n-1} = 0. \tag{3.40}$$

One can of course also use the Miuras to construct the auto-Bäcklund of (3.35). The relevant procedure would be to start from a given x_n and use (3.38*a*) to obtain y_n. Next, since the q-P_{34} is invariant under the transformation $b \to 1/b$, one inverts b in (3.38*b*) and *using the same* y_n one then computes a new x_n, say X_n. (This is put in the proper perspective if one substitutes $m = \sqrt{b/\lambda}$ in (3.38*b*).) The new X_n corresponds to a parameter $A = \lambda^{-10}/a$. Finally, one must use the self-Miura in order to invert the new parameter A. Computing a new x_n, say $\tilde{x}_n$, with (3.36*a*), one obtains the auto-Bäcklund of (3.35) for the new parameter $\tilde{a} = a\lambda^{10}$. Unfortunately, the final expression is too lengthy to be given here explicitly.

4. Conclusions

In this paper, we have presented a collection of results on the various discrete P_{II} equations. While two of these d-P_{II}s had been extensively studied in previous works, for the remaining six known forms no results existed till recently. For all equations analysed, we presented the Miura transformations which made it possible to construct the associated discrete P_{34} (in perfect analogy to what happens in the continuous case).

Several interesting results were obtained along the way. One of the equations examined (3.31), which was initially identified as a discrete P_{II}, turned out to also be a discrete P_{34}, and moreover possessed an auto-Miura (compatible with the two different continuous limits). While analysing the list of known equations we realized that there existed one q-P_{34} for which no corresponding q-P_{II} was known. It turned out that the latter indeed existed and we were able to derive it along with its Miura transformation. For several of the equations we studied, the search for Miura transformations led to what we called self-Miuras which relate the equation to itself, albeit with some changes of parameters and of the independent variable. The usefulness of these self-Miuras becomes evident when one undertakes the construction of the Schlesinger transformations. The standard procedure for this construction is to start from a solution of a d-P_{II}, derive a solution of d-P_{34} through the Miura and use an invariance of the former in order to construct a new solution of d-P_{II}. Usually one obtains an auto-Bäcklund which is not a Schlesinger transformation, i.e. there is no simple relation between the parameters of the initial and final solutions. The way to construct the Schlesinger transformation is to use some invariance of the d-P_{II} (as explained already in the example of the continuous P_{II}) in order to act on the parameters of the solution. However, for many of the d-P_{II}s we examined no such invariance exists. Fortunately in these cases, one can use the self-Miuras and still succeed in constructing the Schlesinger transformations.

For all the d-P_{II}s studied here, we were able to obtain linearizable solutions (which exist for special values of the d-P_{II} parameter). The elementary solutions are given in terms of discrete Airy functions. In most cases, however, the discrete linear equation we obtained had a structure richer than what one would expect for a discrete analogue of the Airy equation. As a matter of fact, the richness of the observed structure is related to the fact that the discrete P_{II} is, in most cases, a reduction of an equation with more degrees of freedom. Curiously enough, for some of the d-P_{II}s we were not able to obtain rational solutions. We cannot tell at this stage whether this absence is the result of some deep property of the equations in question or whether it reflects the inability of the authors to imagine a sufficiently complicated rational solution.

While several discrete analogues of P_{II} have been treated in this paper, bringing the total number to nine, they do not exhaust the list of all possible d-P_{II}s. More examples, based on equations related to affine Weyl groups $E_7^{(1)}$ and $E_8^{(1)}$, certainly exist. We hope to return to this investigation in some future work.

Acknowledgment

ASC acknowledges a CEE scholarship through an RTN contract.

References

[1] Painlevé P 1902 *Acta Math.* **25** 1
[2] Gambier B 1909 *Acta Math.* **33** 1
[3] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095
[4] Ince E L 1956 *Ordinary Differential Equations* (London: Dover)
[5] Okamoto K 1981 *Physica* D **2** 525
[6] Gromak V A and Lukashevich N A 1990 *The Analytic Solutions of the Painlevé Equations* (Minsk: Universitetskoye Publishers) in Russian
[7] Grammaticos B, Ramani A and Carstea A S 2002 Bilinear avatars of the discrete Painlevé II equation *Proceeding of the NATO workshop on Bilinear Integrable Systems* accepted
[8] Ramani A and Grammaticos B 1996 *Physica* A **228** 160
[9] Joshi N, Ramani A and Grammaticos B 1998 *Phys. Lett.* A **249** 59
[10] Ramani A and Grammaticos B 1992 *J. Phys. A: Math. Gen.* **25** L633
[11] Nijhoff F, Satsuma J, Kajiwara K, Grammaticos B and Ramani A 1996 *Inverse Problems* **12** 697
[12] Nijhoff F W 1996 On some 'Schwarzian equations and their discrete analogues' *Algebraic Aspects of Integrable Systems; In Memory of Irene Dorfman* ed A S Fokas and I M Gel'fand (Basle: Birkhäuser) p 237
[13] Ramani A, Grammaticos B, Tamizhmani T and Tamizhmani K M 2000 *J. Phys. A: Math. Gen.* **33** 579
[14] Kruskal M D, Tamizhmani K M, Grammaticos B and Ramani A 2000 *Regul. Chaotic Dyn.* **5** 273